

NON-PERIODIC SYSTEMS WITH CONTINUOUS DIFFRACTION MEASURES

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ABSTRACT. The present state of mathematical diffraction theory for systems with continuous spectral components is reviewed and extended. We begin with a discussion of various characteristic examples with singular or absolutely continuous diffraction, and then continue with a more general exposition of a systematic approach via stationary stochastic point processes. Here, the intensity measure of the Palm measure takes the role of the autocorrelation measure in the traditional approach. We furthermore introduce a ‘Palm-type’ measure for general complex-valued random measures that are stationary and ergodic, and relate its intensity measure to the autocorrelation measure.

1. INTRODUCTION

The (mathematical or kinematic) diffraction theory of systems in Euclidean space with pure point spectrum is rather well understood. Ultimately, this is due to the availability of Poisson’s summation formula and its generalisations to the setting of measures (or to tempered distributions); see [12, Sec. 9.2] for a systematic exposition. Beyond results on the spectral nature, this often also provides explicit formulas for the diffraction measure, such as in the cases of lattice-periodic systems and model sets. For these systems, there is also a well-understood connection with the Halmos–von Neumann theorem for the corresponding pure point dynamical spectrum; see [55, 16, 60, 18, 20] for details as well as [12] and references therein for general background.

As soon as one enters the realm of systems with continuous diffraction spectra (or at least with continuous spectral components), the situation changes drastically. As in the case of Schrödinger operator spectra [31], much less is known about the plethora of possibilities, and there rarely are explicit formulas for the diffraction measures of specific examples. Until recently, explicit results were restricted to simple systems of Bernoulli type (hence with disorder that leads to independent random variables) or to some paradigmatic examples in one dimension (and product systems built from them).

There has now been some progress towards explicitly computable examples in various directions [62, 4, 15]. In particular, both for singular and for absolutely continuous cases, constructive approaches have been more successful than previously anticipated; compare [12, Ch. 10]. Consequently, there is some hope that more systems can be understood in this way. This view is also supported by the recent progress in the understanding of

the connection between the dynamical and the diffraction spectrum in this more general situation; see [20] and references therein. At the same time, such examples will improve our intuition about systems with continuous diffraction. Below, this will be reflected by several short sketches of characteristic examples (which are covered in more detail in [12]), before we embark on a more systematic setting via general point process theory. Our focus is on systems in \mathbb{R}^d , which is the primary situation to understand, particularly from the applications point of view. Extensions to more general locally compact Abelian groups are possible, but will not be discussed here.

2. DIFFRACTION MEASURES — A BRIEF REMINDER

Let ω be a locally finite (and possibly complex) measure on \mathbb{R}^d , which we primarily view as a linear functional on the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support on \mathbb{R}^d , together with some mild extra conditions. In favourable cases, ω will be translation bounded. By the classic Riesz–Markov representation theorem, we may identify the measures defined by this approach with regular Borel measures; for a systematic exposition, we refer to [44, 16] as well as [12, Chs. 8 and 9] and references therein. Particularly important examples comprise the *Dirac measure* δ_x , defined by $\delta_x(g) := g(x)$ for $g \in C_c(\mathbb{R}^d)$, and measures of the form

$$(2.1) \quad \delta_S := \sum_{x \in S} \delta_x,$$

which are known as *Dirac combs*, where $S \subset \mathbb{R}^d$ is uniformly discrete. More generally, we will also consider objects of the form $\sum_{x \in S} w(x) \delta_x$, which can be a measure for a general countable set S , then under suitable conditions on the weight function w . Such measures are referred to as *weighted Dirac combs*.

Recall from [44] or [12] that, if ω is a measure on \mathbb{R}^d , the (inverted-conjugate) measure $\tilde{\omega}$ is defined by $\tilde{\omega}(g) := \overline{\omega(\tilde{g})}$ for $g \in C_c(\mathbb{R}^d)$, where $\tilde{g}(x) := \overline{g(-x)}$. Given a measure ω , consider its *autocorrelation measure*

$$(2.2) \quad \gamma = \gamma_\omega := \omega \circledast \tilde{\omega},$$

where \circledast denotes the volume averaged (or Eberlein) convolution. The latter is defined by

$$\omega \circledast \tilde{\omega} := \lim_{r \rightarrow \infty} \frac{\omega_r * \tilde{\omega}_r}{\text{vol}(B_r(0))}$$

with $B_r(0)$ the (open) ball of radius r around the origin and $\omega_r := \omega|_{B_r(0)}$. At this stage, we assume the existence of the limit. This will be discussed in more detail later.

If (as in many of our examples) ω is a Dirac comb with lattice support, also γ will be supported on the same lattice (or a subset of it). Concretely, if

$$\omega = w \delta_{\mathbb{Z}} := \sum_{n \in \mathbb{Z}} w(n) \delta_n,$$

with a bounded weight function w say, one finds $\gamma = \eta\delta_{\mathbb{Z}}$ with the positive definite function $\eta: \mathbb{Z} \rightarrow \mathbb{C}$ being defined by

$$\begin{aligned}
 \eta(m) &:= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N w(n) \overline{w(n-m)} \\
 (2.3) \qquad &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \overline{w(n)} w(n+m),
 \end{aligned}$$

provided that all limits exist. In our exposition below, this existence will follow by suitable applications of Birkhoff's ergodic theorem, applied to the dynamical system of the shift action on the orbit closure of the sequence w or to a similar type of dynamical system; compare [16] for a more general setting. One benefit of this approach will emerge via the Herglotz–Bochner theorem [50].

The autocorrelation measure γ is *positive definite* (or of positive type) by construction, which means that $\gamma(g * \tilde{g}) \geq 0$ for all $g \in C_c(\mathbb{R}^d)$. It is thus Fourier transformable [27], and the Fourier transform $\hat{\gamma}$ is a positive measure, called the *diffraction measure* of ω ; see [29] for the physics behind this notion, and [44] as well as [12, Ch. 9] for the mathematical theory. Within the framework of kinematic diffraction, it describes the outcome of a scattering experiment by quantifying how much intensity is scattered into a given volume of d -space, and thus is the central object of our interest. By the Lebesgue decomposition theorem, there is a unique splitting

$$\hat{\gamma} = \hat{\gamma}_{\text{pp}} + \hat{\gamma}_{\text{sc}} + \hat{\gamma}_{\text{ac}}$$

of the diffraction measure into its *pure point* part $\hat{\gamma}_{\text{pp}}$, its *singular continuous* part $\hat{\gamma}_{\text{sc}}$ and its *absolutely continuous* part $\hat{\gamma}_{\text{ac}}$, with respect to Lebesgue measure λ . The pure point part comprises the ‘Bragg peaks’ (of which there are at most countably many, so $\hat{\gamma}_{\text{pp}}$ is a sum over at most countably many Dirac measures with positive weights), while the absolutely continuous part corresponds to the diffuse ‘background’ scattering which is given by a locally integrable density relative to λ . The singular continuous part is whatever remains — if present, it is a measure that gives no weight to single points, but is still concentrated to an (uncountable) set of zero Lebesgue measure.

Measures ω which lead to a diffraction $\hat{\gamma} = \hat{\gamma}_{\text{pp}}$ are called *pure point diffractive*; examples include lattice-periodic measures and measures based on model sets. These have been studied in detail in the context of diffraction of crystals and quasicrystals; see [9] for a recent review and [12, Chs. 8 and 9] for a systematic exposition. Here, we are concentrating on the other two spectral components, which may also carry important information on the (partial) order which is present in the underlying structure. Pure point spectra are discussed in detail in [23, 16, 17, 18, 12, 59, 57, 58]; for related spectral problems in the context of Schrödinger operators, we refer to [31].

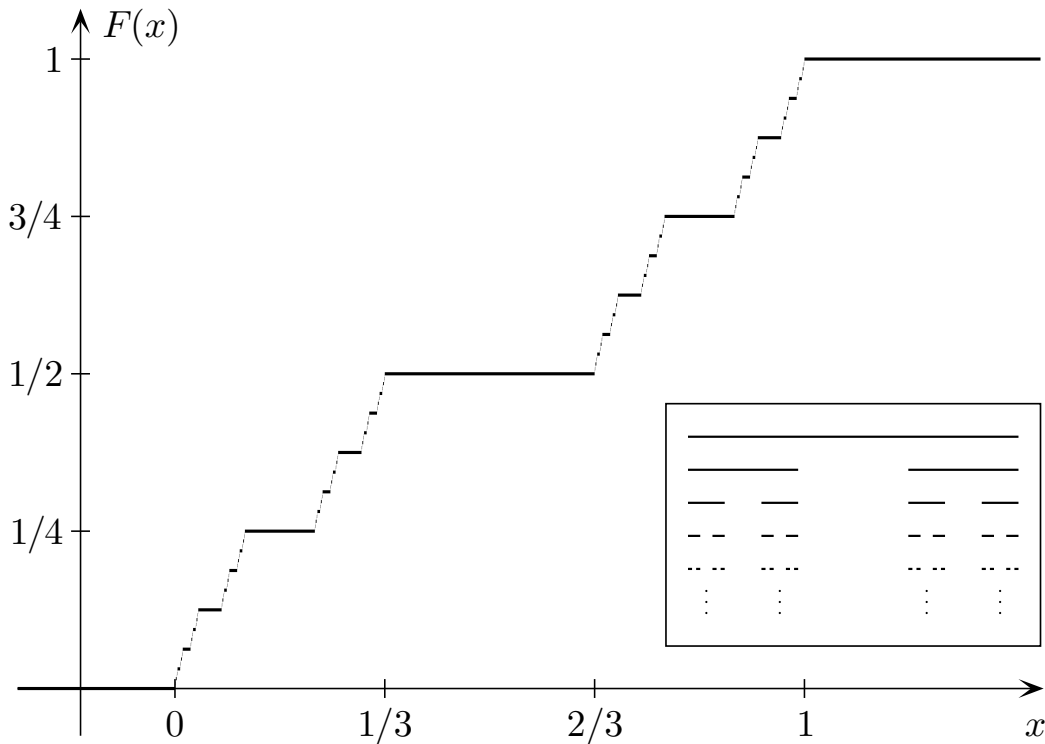


FIGURE 1. The distribution function F of the classic middle-thirds Cantor measure. The construction of the underlying Cantor set is sketched in the inset.

3. GUIDING EXAMPLES

As mentioned above, the understanding of systems with continuous diffraction components is less developed than that of pure point diffractive ones. Still, a better intuition will emerge from a sample of characteristic examples. It is the purpose of this section to provide some of them, while we refer to the literature for further ones [13, 22, 4, 9, 15, 12].

3.1. Thue–Morse Sequences. Let us begin with a classic example from the theory of substitution systems that leads to a singular continuous diffraction measure with rather different features in comparison with the Cantor measure, the latter being illustrated in Figure 1. Our example has a long history, which can be extracted from [79, 61, 47, 1]. We confine ourselves to a brief summary of the results, and refer to [7, 12] and references therein for proofs and details.

The classic *Thue–Morse* (TM) sequence can be defined via the one-sided fixed point $v = v_0 v_1 v_2 \dots$ (with $v_0 = 1$) of the primitive substitution rule

$$\varrho: \begin{array}{l} 1 \mapsto 1\bar{1} \\ \bar{1} \mapsto \bar{1}1 \end{array}$$

on the binary alphabet $\{1, \bar{1}\}$. The fixed point is the limit (in the obvious product topology) of the (suitably embedded) iteration sequence

$$1 \xrightarrow{\varrho} 1\bar{1} \xrightarrow{\varrho} 1\bar{1}\bar{1}1 \xrightarrow{\varrho} 1\bar{1}\bar{1}1\bar{1}11\bar{1} \xrightarrow{\varrho} \dots \longrightarrow v = \varrho(v) = v_0v_1v_2v_3\dots$$

and has a number of distinctive properties [1, 68], for instance

- $v_i = (-1)^{\text{sum of the binary digits of } i}$
- $v_{2i} = v_i$ and $v_{2i+1} = \bar{v}_i$, for all $i \in \mathbb{N}_0$;
- $v = v_0v_2v_4\dots$ and $\bar{v} = v_1v_3v_5\dots$
- v is (strongly) cube-free (and hence non-periodic).

Here, we define $\bar{\bar{1}} = 1$ and identify $\bar{1}$ with -1 , also for the later calculations with Dirac combs. A two-sided sequence w can be defined by

$$w(i) = \begin{cases} v_i, & \text{for } i \geq 0, \\ v_{-i-1}, & \text{for } i < 0, \end{cases}$$

which is a fixed point of ϱ^2 , because the seed $w_{-1}|w_0 = 1|1$ is a legal word (it occurs in $\varrho^3(1)$) and $w = \varrho^2(w)$. The (discrete) hull $\mathbb{X} = \mathbb{X}_{\text{TM}}$ of the TM substitution is the closure of the orbit of w under the shift action, which is a subset of $\{\pm 1\}^{\mathbb{Z}}$ and hence a compact space. The orbit of any of its members is dense in \mathbb{X} . We thus have a topological dynamical system (\mathbb{X}, \mathbb{Z}) that is minimal. When equipped with the standard Borel σ -algebra, the system admits a unique shift-invariant probability measure ν , so that the corresponding measure theoretic dynamical system $(\mathbb{X}, \mathbb{Z}, \nu)$ is strictly ergodic [47, 68].

Any given $w \in \mathbb{X}$ is mapped to a signed Dirac comb ω via

$$\omega = \sum_{n \in \mathbb{Z}} w(n) \delta_n.$$

The image of \mathbb{X} is a space of translation bounded measures that is compact in the vague topology. We inherit strict ergodicity via conjugacy, and thus obtain an autocorrelation of the form of Eq. (2.2) with coefficients $\eta(m)$ as in Eq. (2.3). In fact, this autocorrelation does not depend on the choice of the element from \mathbb{X} , so that we may choose the fixed point w from above for the concrete analysis. Due to the nature of w , the coefficients can alternatively be expressed as

$$\eta(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n v_{n+m}$$

for $m \geq 0$, together with $\eta(-m) = \eta(m)$. It is clear that $\eta(0) = 1$, and the scaling relations of v lead to the recursions [47]

$$(3.1) \quad \begin{aligned} \eta(2m) &= \eta(m) & \text{and} \\ \eta(2m+1) &= -\frac{1}{2}(\eta(m) + \eta(m+1)), \end{aligned}$$

which are valid for all $m \in \mathbb{Z}$. In particular, the second relation, used with $m = 0$, implies $\eta(1) = -\frac{1}{3}$, which can also be calculated directly.

Since $\eta : \mathbb{Z} \rightarrow \mathbb{C}$ is a positive definite function with $\eta(0) = 1$, there is a unique probability measure μ on the unit circle (which we identify with the unit interval here) such that

$$(3.2) \quad \eta(m) = \int_0^1 e^{2\pi i m y} d\mu(y),$$

which is a consequence of the Herglotz–Bochner theorem [50, Thm. I.7.6]. Since ω is supported on \mathbb{Z} , the corresponding diffraction measure $\hat{\gamma}$ is 1-periodic, which follows from [3, Thm. 1]; see also [12, Sec. 10.3.2]. One then finds the relation

$$\hat{\gamma} = \mu * \delta_{\mathbb{Z}}$$

with the measure μ from Eq. (3.2), appropriately interpreted as a measure on $[0, 1)$ and hence also on \mathbb{R} . Clearly, one also has $\mu = \hat{\gamma}|_{[0,1)}$. One can now analyse the spectral type of $\hat{\gamma}$ via that of the finite measure μ , where we follow [47]; see also [68, 20].

Defining $\Sigma(N) = \sum_{m=-N}^N (\eta(m))^2$, a two-step calculation with the recursion (3.1) establishes the inequality $\Sigma(4N) \leq \frac{3}{2}\Sigma(2N)$ for all $N \in \mathbb{N}$. This implies $\lim_{N \rightarrow \infty} \Sigma(N)/N = 0$, wherefore Wiener’s criterion [79, 50], see also [12, Prop. 8.9], tells us that μ is a continuous measure, so that $\hat{\gamma}$ cannot have any pure point component. Note that the absence of the ‘trivial’ pure point component of $\hat{\gamma}$ on \mathbb{Z} is due to the use of balanced weights, in the sense that 1 and -1 are equally frequent. Consequently, the average weight is zero, and the claim follows from [12, Prop. 9.2].

Let us now define the distribution function F by $F(x) = \mu([0, x])$ for $x \in [0, 1]$, which is a continuous function that defines a Riemann–Stieltjes measure [53, Ch. X], so that $dF = \mu$. The recursion relation for η now implies [47] the two functional relations

$$dF\left(\frac{x}{2}\right) \pm dF\left(\frac{x+1}{2}\right) = \left\{ -\cos(\pi x) \right\} dF(x),$$

which have to be satisfied by the **ac** and **sc** parts of F separately, because we have $\mu_{\text{ac}} \perp \mu_{\text{sc}}$ in the measure-theoretic sense; see [72, Thm. I.20] or [53, Thm. VII.2.4]. Therefore, defining

$$\eta_{\text{ac}}(m) = \int_0^1 e^{2\pi i m x} dF_{\text{ac}}(x),$$

we know that the coefficients $\eta_{\text{ac}}(m)$ must satisfy the same recursions (3.1) as $\eta(m)$, possibly with a different initial condition $\eta_{\text{ac}}(0)$. The classic Riemann–Lebesgue lemma [50, Thm. I.2.8] states that $\lim_{m \rightarrow \pm\infty} \eta_{\text{ac}}(m) = 0$. But this limit is only compatible with $\eta_{\text{ac}}(0) = 0$, because $\eta_{\text{ac}}(1) = -\frac{1}{3}\eta_{\text{ac}}(0)$ and $\eta_{\text{ac}}(2m) = \eta_{\text{ac}}(m)$ for all $m \in \mathbb{N}$, so that we must have $\eta_{\text{ac}} \equiv 0$. This means $F_{\text{ac}} = 0$ by the Fourier uniqueness theorem, wherefore μ and hence $\hat{\gamma}$ (neither of which is the zero measure) are purely singular continuous. The resulting distribution function F is illustrated in Figure 2. Note that F can consistently be extended to a continuous function on \mathbb{R} via $F(x+n) = F(x) + n$ for $n \in \mathbb{Z}$ and then

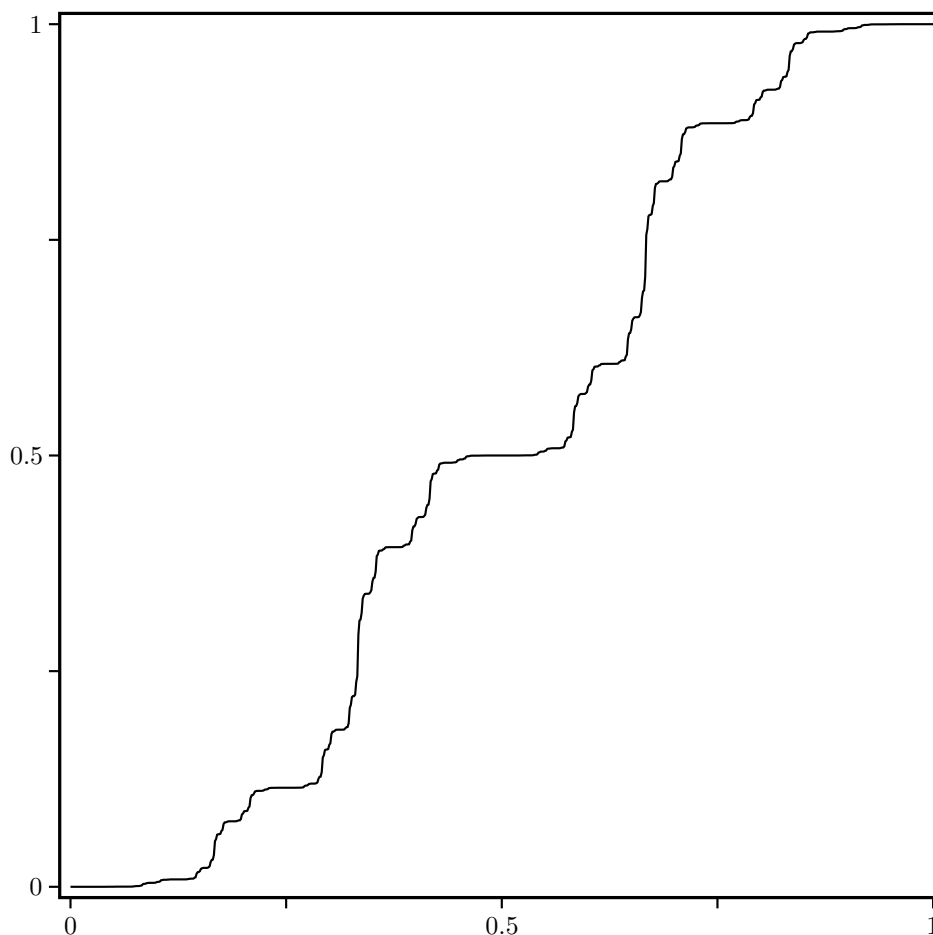


FIGURE 2. The strictly increasing distribution function of the classic, purely singular continuous TM measure on $[0, 1]$.

defines $\widehat{\gamma}$ via $dF = \widehat{\gamma}$ in the Lebesgue–Stieltjes sense. The function F can efficiently be calculated by means of the uniformly converging Volterra iteration

$$(3.3) \quad F_{n+1}(x) = \frac{1}{2} \int_0^{2x} (1 - \cos(\pi y)) F_n'(y) dy$$

with $F_0(x) = x$. In contrast to the Devil’s staircase of Figure 1, the TM distribution function is *strictly* increasing, which means that there is no plateau (which would indicate a gap in the support of $\widehat{\gamma}$); see [7, 12] and references therein for details and further properties of F . So far, we have obtained the following result.

Theorem 3.1. *Let w be any element of the Thue–Morse hull $\mathbb{X} = \mathbb{X}_{\text{TM}}$, the latter represented as a closed subshift of $\{\pm 1\}^{\mathbb{Z}}$, and consider the corresponding Dirac comb $w\delta_{\mathbb{Z}}$. Then, its autocorrelation γ exists and is given by $\gamma = \eta\delta_{\mathbb{Z}}$ with η being defined by Eq. (3.1) together with the initial condition $\eta(0) = 1$.*

The diffraction measure is $\widehat{\gamma} = \mu * \delta_{\mathbb{Z}}$, where μ is the purely singular continuous probability measure from Eq. (3.2). In particular, $\widehat{\gamma}$ is purely singular continuous as well. \square

To go one step further, Eq. (3.3) defines an iteration sequence of distribution functions for absolutely continuous measures that converges towards the TM measure in the vague topology. Writing $dF_n(x) = f_n(x) dx$, one finds

$$f_n(x) = \prod_{m=0}^{n-1} (1 - \cos(2^{m+1}\pi x)),$$

which, in the vague limit as $n \rightarrow \infty$, gives the well-known Riesz product representation of the TM measure; compare [68] for details and [80] for general background on Riesz products.

The TM sequence is closely related to the limit-periodic *period doubling* (pd) sequence, compare [5, 12] and references therein, via the (continuous) sliding block map defined by

$$(3.4) \quad \phi: \quad 1\bar{1}, \bar{1}1 \mapsto a, \quad 11, \bar{1}\bar{1} \mapsto b,$$

which results in an exact 2-to-1 surjection from the hull \mathbb{X}_{TM} to \mathbb{X}_{pd} . The latter is the hull of the period doubling substitution defined by

$$(3.5) \quad \varrho_{\text{pd}}: \quad a \mapsto ab, \quad b \mapsto aa.$$

Viewed as topological dynamical systems, this means that $(\mathbb{X}_{\text{pd}}, \mathbb{Z})$ is a factor of $(\mathbb{X}_{\text{TM}}, \mathbb{Z})$. Since both are strictly ergodic, this extends to the corresponding measure-theoretic dynamical systems. The period doubling sequence can be described as a regular model set with a 2-adic internal space [24, 23] and is thus pure point diffractive. This pairing also explains a phenomenon observed in [78], namely that the *dynamical* spectrum of the TM system is richer than its diffraction spectrum. By the dynamical (or von Neumann) spectrum, we mean the spectrum of the unitary operator induced by the shift on the Hilbert space $L^2(\mathbb{X}, \nu)$, where ν is the unique shift-invariant probability measure on \mathbb{X} ; see [68] for more. Here, the pure point part of the dynamical spectrum is the ring $\mathbb{Z}[\frac{1}{2}]$, which is not even finitely generated (and only the ‘trivial’ part \mathbb{Z} is detected by the diffraction measure of the TM system with general weights). In fact, our above measure μ from Theorem 3.1 represents the maximal spectral measure in the ortho-complement of the pure point sector [68, 20]. The missing pure point part, however, is fully recovered via the *diffraction* measure of \mathbb{X}_{pd} ; see [12] for details and [20] for a general discussion of this phenomenon.

Various generalisations of this result are known by now. First of all, and perhaps not surprisingly, this generalises to an entire family of bijective, binary substitutions [5]. Moreover, extensions to higher dimensions are also possible, including the explicit nature of the resulting diffraction measure; compare [40, 11] and references therein.

3.2. Rudin–Shapiro Sequence. The (binary) Rudin–Shapiro (RS) chain is a bi-infinite deterministic sequence, with polynomial (in fact linear) complexity function and thus zero entropy. It can be described recursively as $w = (w(n))_{n \in \mathbb{Z}}$ with $w(n) \in \{\pm 1\}$, with initial conditions $w(-1) = -1$, $w(0) = 1$ and the recursion

$$(3.6) \quad w(4n + \ell) = \begin{cases} w(n), & \text{for } \ell \in \{0, 1\}, \\ (-1)^{n+\ell} w(n), & \text{for } \ell \in \{2, 3\}, \end{cases}$$

which determines $w(n)$ for all $n \in \mathbb{Z}$. The orbit closure of w under the shift action is the (discrete) RS hull \mathbb{X}_{RS} . Alternatively, one can start from a primitive substitution on a 4-letter alphabet (via $a \mapsto ac$, $b \mapsto dc$, $c \mapsto ab$ and $d \mapsto db$) and define a quaternary hull, which then maps to the binary hull via a simple reduction to two letters (for instance via $a, c \mapsto 1$ and $b, d \mapsto -1$); compare [1, 68] or [12, Sec. 4.7.1] for details. The two hulls define topologically conjugate dynamical systems, with local derivation rules in both directions; see [12, Rem. 4.11].

The shift action on \mathbb{X}_{RS} is strictly ergodic, so that one can define functions $\eta, \vartheta: \mathbb{Z} \rightarrow \mathbb{C}$ via

$$\begin{aligned} \eta(m) &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N w(n) w(n-m) \quad \text{and} \\ \vartheta(m) &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N (-1)^n w(n) w(n-m), \end{aligned}$$

where all limits exist due to unique ergodicity (which is best formulated on the level of the 4-letter alphabet mentioned above). In particular, one finds $\eta(0) = 1$ and $\vartheta(0) = 0$. The recursive structure of Eq. (3.6) now implies the validity of a closed set of recursive equations [8, 9], namely

$$\begin{aligned} \eta(4m) &= \frac{1+(-1)^m}{2} \eta(m), \\ \eta(4m+1) &= \frac{1-(-1)^m}{4} \eta(m) + \frac{(-1)^m}{4} \vartheta(m) - \frac{1}{4} \vartheta(m+1), \\ \eta(4m+2) &= 0, \\ \eta(4m+3) &= \frac{1+(-1)^m}{4} \eta(m+1) - \frac{(-1)^m}{4} \vartheta(m) + \frac{1}{4} \vartheta(m+1), \end{aligned}$$

together with

$$\begin{aligned} \vartheta(4m) &= 0, \\ \vartheta(4m+1) &= \frac{1-(-1)^m}{4} \eta(m) - \frac{(-1)^m}{4} \vartheta(m) + \frac{1}{4} \vartheta(m+1), \\ \vartheta(4m+2) &= \frac{(-1)^m}{2} \vartheta(m) + \frac{1}{2} \vartheta(m+1), \\ \vartheta(4m+3) &= -\frac{1+(-1)^m}{4} \eta(m+1) - \frac{(-1)^m}{4} \vartheta(m) + \frac{1}{4} \vartheta(m+1). \end{aligned}$$

which hold for all $m \in \mathbb{Z}$; see [12, Sec. 10.2] for details. A careful inspection shows that the unique solution of this set of equations, with the initial conditions mentioned above, is $\eta(m) = \delta_{m,0}$ together with $\vartheta(m) = 0$ for all $m \in \mathbb{Z}$. Hence, despite the deterministic nature of the RS sequence, the autocorrelation measure is simply given by $\gamma_{\text{RS}} = \delta_0$, so that $\widehat{\gamma_{\text{RS}}} = \lambda$, where λ again denotes Lebesgue measure. Alternatively, the result also follows from the exposition in [68, 67].

Theorem 3.2. *Let w be any element of the Rudin–Shapiro hull $\mathbb{X}_{\text{RS}} \subset \{\pm 1\}^{\mathbb{Z}}$, and consider the corresponding Dirac comb $w\delta_{\mathbb{Z}}$. Then, its autocorrelation exists and is given by $\gamma_{\text{RS}} = \delta_0$, with diffraction measure $\widehat{\gamma_{\text{RS}}} = \lambda$. \square*

As in the case of the TM sequence, the non-trivial pure point part of the dynamical spectrum (which is $\mathbb{Z}[\frac{1}{2}]$ once again) is not ‘seen’ by the diffraction measure, while λ (with multiplicity 2) represents once again the maximal spectral measure in the orthocomplement of the pure point sector. However, the missing pure point component can be recovered by a suitable factor system, the latter obtained via the block map defined by Eq. (3.4). The corresponding factor is represented by a limit-periodic substitution rule that is somewhat reminiscent of the paper folding sequence [1]; see [12, Sec. 10.2] for a complete discussion and [20] for the general connection between dynamical and diffraction spectra. The structure underlying the RS sequence can be generalised to higher-dimensional lattice substitutions in a rather systematic way; see [39] for details.

3.3. Bernoullisation. Let us begin this discussion by recalling the structure of the full Bernoulli shift from the viewpoint of kinematic diffraction. The classic coin tossing process leads to the Dirac comb

$$\omega = \sum_{n \in \mathbb{Z}} X(n) \delta_n,$$

where the $(X(n))_{n \in \mathbb{Z}}$ form an i.i.d. family of random variables, each taking values 1 and -1 with probabilities p and $1 - p$, respectively. By an application of the strong law of large numbers (SLLN, see [36] for a favourable formulation), almost every realisation has the autocorrelation measure

$$\gamma = (2p - 1)^2 \delta_{\mathbb{Z}} + 4p(1 - p) \delta_0,$$

and hence (via Fourier transform) the diffraction measure

$$\widehat{\gamma} = (2p - 1)^2 \delta_{\mathbb{Z}} + 4p(1 - p) \lambda.$$

Here, we have used the classic Poisson summation formula $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$; compare [9] and references therein, as well as [12, Sec. 9.2] for a formulation in the diffraction context. When $p = \frac{1}{2}$, the diffraction boils down to $\widehat{\gamma} = \lambda$. Here, the point part is extinct because the average scattering strength vanishes. For proofs, we refer the reader to [22, 4], while [51, 52] contain several important and non-trivial generalisations and extensions; see also [56] for important related material.

The Bernoulli chain has (metric) entropy [28, 35]

$$h(p) = -p \log(p) - (1-p) \log(1-p),$$

which is maximal for $p = \frac{1}{2}$, with $h(\frac{1}{2}) = \log(2)$. It vanishes for the deterministic limiting cases $p \in \{0, 1\}$. For the latter, we have $\omega = \mp \delta_{\mathbb{Z}}$, and consequently obtain the pure point diffraction measure $\widehat{\gamma} = \delta_{\mathbb{Z}}$, again via Poisson's summation formula.

Now, the theory of random variables allows for an interpolation between deterministic (binary) sequences and coin tossing sequences as follows. If $w \in \{\pm 1\}^{\mathbb{Z}}$ denotes a deterministic sequence (which we assume to be uniquely ergodic for simplicity), consider the random Dirac comb [8]

$$(3.7) \quad \omega_p = \sum_{n \in \mathbb{Z}} w(n) X(n) \delta_n,$$

where $(X(n))_{n \in \mathbb{Z}}$ is, as above, an i.i.d. family of random variables with values in $\{\pm 1\}$ and probabilities p and $1 - p$. This ‘Bernoullisation’ of w can be viewed as a ‘model of second thoughts’, where the sign of the weight at position n is changed with probability $1 - p$; compare [12, Sec. 11.2.2].

Let w now be the Rudin–Shapiro sequence from above. By a (slightly more complicated) application of the SLLN, it can be shown [8] that the autocorrelation γ_p of the Dirac comb ω_p is then almost surely given by

$$\gamma_p = (2p - 1)^2 \gamma_{\text{RS}} + 4p(1 - p) \delta_0 = \delta_0,$$

irrespective of the value of the parameter $p \in [0, 1]$. Recall that two measures with the same autocorrelation are called *homometric*; see [12, Sec. 9.6] for background. Our observation thus establishes the following classic result; see [8, 9, 12] for details.

Theorem 3.3. *The random Dirac combs ω_p of Equation (3.7) with real parameter values $p \in [0, 1]$ are (almost surely) homometric, with absolutely continuous diffraction measure $\widehat{\gamma}_p = \widehat{\gamma}_{\text{RS}} = \lambda$, irrespective of the value of p . In other words, the family $\{\omega_p \mid p \in [0, 1]\}$ is (almost surely) isospectral. \square*

This result shows that diffraction can be insensitive to entropy, because the family of Dirac combs ω_p of Eq. (3.7) continuously interpolates between the deterministic Rudin–Shapiro case with zero entropy and the completely random Bernoulli chain with maximal entropy $\log(2)$. Clearly, the Bernoullisation procedure can be applied to other sequences as well, and can be generalised to higher dimensions. For further aspects of entropy versus diffraction, we refer to [8, 10, 19].

3.4. Random Dimers on the Line. Another instructive example [25] is based on certain dimer configurations on \mathbb{Z} . To formulate it, we follow the exposition in [10] and partition \mathbb{Z} into a close-packed arrangement of ‘dimers’ (pairs of neighbours), without gaps or overlaps. Clearly, there are just two possibilities to do so, because the position of the first dimer fixes

that of all others. Next, decorate each dimer randomly with either $(1, -1)$ or $(-1, 1)$, with equal probability. This results in patches such as

$$\begin{aligned} & \dots [+ -][- +][- +][+ -][- +][- +][- +][+ -][+ -] \dots \\ & \dots [- +][+ -][+ -][- +][+ -][+ -][+ -][- +][- +][+ -] \dots \end{aligned}$$

where the dimer boxes are indicated by brackets. The set of all decorated sequences defined in this way is given by

$$\mathbb{X} = \{w \in \{\pm 1\}^{\mathbb{Z}} \mid M(w) \subset 2\mathbb{Z} \text{ or } M(w) \subset 2\mathbb{Z} + 1\},$$

where $M(w) := \{n \in \mathbb{Z} \mid w(n) = w(n+1)\}$. Note that $M(w)$ is empty precisely for the two periodic sequences that are defined by $w(n) = \pm(-1)^n$ for $n \in \mathbb{Z}$. Clearly, $\mathbb{X} \subset \{\pm 1\}^{\mathbb{Z}}$ is closed and hence compact.

Let $w \in \mathbb{X}$ and consider the corresponding signed Dirac comb on \mathbb{Z} with binary weights $w(n) \in \{\pm 1\}$. One can then show (again via the SLLN) that the corresponding autocorrelation almost surely exists and is given by [25]

$$(3.8) \quad \gamma = \delta_0 - \frac{1}{2}(\delta_1 + \delta_{-1}).$$

The corresponding diffraction measure is then

$$(3.9) \quad \widehat{\gamma} = (1 - \cos(2\pi k))\lambda,$$

which is again purely absolutely continuous. Here, the (smooth) Radon–Nikodym density relative to λ is written as a function of k . Note that the diffraction measure for general weights h_+ and h_- is given by

$$\widehat{\gamma}_{\pm} = \frac{|h_+ + h_-|^2}{4} \delta_{\mathbb{Z}} + \frac{|h_+ - h_-|^2}{4} \widehat{\gamma}$$

with $\widehat{\gamma}$ as in Eq. (3.9). In particular, the measure $\widehat{\gamma}_{\pm}$ shows only the ‘trivial’ pure point diffraction contribution that arises as the consequence of \mathbb{Z} being the support of the weighted measure under consideration. The same phenomenon also occurs for general (non-balanced) TM and RS sequences; compare [12, Rems. 10.3 and 10.5].

On first sight, the system looks disordered, with entropy $\frac{1}{2} \log(2)$. This seems (qualitatively) reflected by the diffraction. However, the system also defines a measure-theoretic dynamical system under the action of \mathbb{Z} , as generated by the shift. As such, it has a dynamical spectrum that does contain a pure point part, with eigenvalues 0 and $\frac{1}{2}$; we refer to [68] for general background on this concept, and to [25] for the actual calculation of the eigenfunctions. The extension to a (continuous) dynamical system $\mathbb{X}_{\mathbb{C}}$ under the general translation action of \mathbb{R} is done via suspension; see [28, Ch. 11.1] (where the suspension is called a special flow) or [35] for general background.

This finding suggests that some degree of order must be present that is neither visible from the entropy calculation nor from the diffraction measure alone. Indeed, in analogy with the situation of the TM and the RS sequence, one can define a factor of the system

by a sliding block map $\phi: \mathbb{X} \longrightarrow \{\pm 1\}^{\mathbb{Z}}$ defined by $(\phi w)(n) = -w(n)w(n+1)$. It maps \mathbb{X} globally 2:1 onto

$$\mathbb{Y} = \phi(\mathbb{X}) = \{v \in \{\pm 1\}^{\mathbb{Z}} \mid v(n) = 1 \text{ for all } n \in 2\mathbb{Z} \text{ or for all } n \in 2\mathbb{Z} + 1\}.$$

The suspension \mathbb{Y}_c (for the action of \mathbb{R}) is defined as above. The mapping ϕ extends accordingly.

The autocorrelation and diffraction measures of the signed Dirac comb $v\delta_{\mathbb{Z}}$ for an element $v \in \mathbb{Y}$ are almost surely given by

$$\gamma = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{2\mathbb{Z}} \quad \text{and} \quad \hat{\gamma} = \frac{1}{2}\lambda + \frac{1}{4}\delta_{\mathbb{Z}/2}.$$

The diffraction of the factor system \mathbb{Y} uncovers the ‘hidden’ pure point part of the dynamical spectrum, which was absent in the purely absolutely continuous diffraction of the signed Dirac comb $w\delta_{\mathbb{Z}}$ with $w \in \mathbb{X}$. In summary, we have the following situation [25, 20].

Theorem 3.4. *The diffraction measure of the close-packed dimer system \mathbb{X} with balanced weights is purely absolutely continuous and given by Eq. (3.9), which holds almost surely relative to the natural invariant measure of the system.*

The dynamical spectrum of the continuous close-packed dimer system \mathbb{X}_c under the translation action of \mathbb{R} contains the pure point part $\mathbb{Z}/2$ together with a countable Lebesgue spectrum.

The non-trivial part $\mathbb{Z} + \frac{1}{2}$ of the dynamical point spectrum is not reflected by the diffraction spectrum of \mathbb{X}_c , but can be recovered via the diffraction spectrum of a suitable factor, such as \mathbb{Y}_c . \square

As in the case of the Thue–Morse system, where the missing pure point part of the dynamical spectrum is recovered by the diffraction measure of the period doubling factor, we thus see that and how we can recover the missing eigenvalue via a generalised 2-point function. This observation can be extended to symbolic systems over finite alphabets and also to uniquely ergodic Delone dynamical systems of finite local complexity; see [20] for details.

3.5. Ledrappier’s Shift Space. For a long time, people had expected that higher dimensions are perhaps more difficult, but not substantially different. This turned out to be a false premise though, as can be seen from the now classic monograph [73].

In our present context, we pick one characteristic example, the system due to Ledrappier [54], to demonstrate a new phenomenon. We follow the brief exposition in [10] and consider a specific subset of the full shift space $\{\pm 1\}^{\mathbb{Z}^2}$, defined by

$$(3.10) \quad \mathbb{X}_L = \{w \in \{\pm 1\}^{\mathbb{Z}^2} \mid w(x)w(x+e_1)w(x+e_2) = 1 \text{ for all } x \in \mathbb{Z}^2\},$$

where e_1 and e_2 denote the standard Euclidean basis vectors in the plane. On top of being a closed subshift, \mathbb{X}_L is also an Abelian group (here written multiplicatively), which then

comes with a unique, normalised Haar measure. The latter is also shift invariant, and the most natural measure to be considered in our context; see also the reformulation in terms of Gibbs (or equilibrium) measures in [74].

The system is interesting because the number of patches of a given radius (up to translations) grows exponentially in the radius rather than in the area of the patch. This phenomenon is called *entropy of rank 1*, and indicates a new class of systems in higher dimensions. More precisely, along any lattice direction of \mathbb{Z}^2 , the linear subsystems essentially behave like one-dimensional Bernoulli chains. It is thus not too surprising that the diffraction measure satisfies the following theorem, though its proof [26] has to take care of the special directions connected with the defining relations of \mathbb{X}_L .

Theorem 3.5. *If w is an element of the Ledrappier subshift \mathbb{X}_L of Eq. (3.10), the corresponding weighted Dirac comb $w\delta_{\mathbb{Z}^2}$ has diffraction measure λ , which holds almost surely relative to the Haar measure of \mathbb{X}_L . \square*

So, the Ledrappier system is homometric to the (full) Bernoulli shift on $\{\pm 1\}^{\mathbb{Z}^2}$, which means that an element of either system almost surely has diffraction measure λ . As mentioned before, via a suitable product of two Rudin–Shapiro chains, also a deterministic system with diffraction λ exists. This clearly demonstrates the insensitivity of pair correlations to the (entropic) type of order or disorder in the underlying system; see also [8]. Due to the defining relation in Eq. (3.10), it is clear that certain three-point correlations in the Ledrappier system cannot vanish, and thus make it distinguishable from the Bernoulli shift.

Although correlation functions of third order can resolve the situation in this case (and in many other examples as well [32, 57]), one can consider other dynamical systems (such as the $(\times 2, \times 3)$ -shift [26]) that share almost all correlation functions with the Bernoulli shift on $[0, 1]^{\mathbb{Z}^2}$. This is a clear indication that our present understanding of ‘order’ is incomplete, and that we still lack a good set of tools for the detection and classification of order. For a recent alternative based on direct space statistics, we refer to [6].

3.6. Random Matrix Ensembles. Another interesting class of random point sets derives from the (scaled) eigenvalue distribution of certain random matrix ensembles; see [14] and references therein. The global eigenvalue distribution of random orthogonal, unitary or symplectic matrix ensembles is known to asymptotically follow the classic semi-circle law. More precisely, this law describes the eigenvalue distribution of the underlying ensembles of symmetric, Hermitian or (symplectically) self-dual matrices with Gaussian distributed entries. The corresponding random matrix ensembles are called GOE, GUE and GSE, with attached β -parameters 1, 2 and 4, respectively. They permit an interpretation as a Coulomb gas, where β is the power in the central potential; see [2, 63] for general background and [34, 38] for the results that are relevant here.

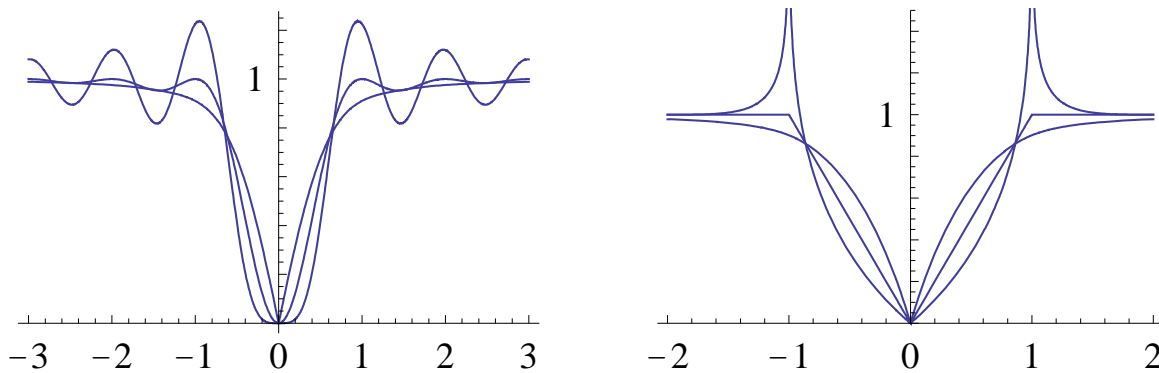


FIGURE 3. Absolutely continuous part of the autocorrelation (left) and the diffraction (right) for the three random matrix derived point set ensembles on the line, with $\beta \in \{1, 2, 4\}$. On the left, the oscillatory behaviour increases with β . On the right, $\beta = 2$ corresponds to the piecewise linear function with bends at 0 and ± 1 , while $\beta = 4$ shows a locally integrable singularity at ± 1 . The latter reflects the slowly decaying oscillations on the left.

For matrices of dimension N , the semi-circle has radius $\sqrt{2N/\pi}$ and area N . Note that, in comparison with [63], we have rescaled the density by a factor $1/\sqrt{\pi}$, so that we really have a semi-circle (and not a semi-ellipse). To study the local eigenvalue distribution for diffraction, we rescale the central region (between ± 1 , say) by $\sqrt{2N/\pi}$. This leads, in the limit as $N \rightarrow \infty$, to an ensemble of point sets on the line that can be interpreted as a stationary, ergodic point process of intensity 1; for $\beta = 2$, see [2, Ch. 4.2] and references therein for details. Since the underlying process is simple (meaning that, almost surely, no point is occupied twice), almost all realisations are point sets of density 1.

It is possible to calculate the autocorrelation of these processes, on the basis of Dyson's correlation functions [34]. Though these functions originally apply to the circular ensembles, they have been adapted to the other ensembles by Mehta [63]. For all three ensembles mentioned above, this leads to an autocorrelation of the form

$$(3.11) \quad \gamma = \delta_0 + (1 - f(|x|))\lambda$$

where f is a locally integrable function that depends on β ; see [14] for the explicit formulas, and the left panel of Figure 3 for an illustration.

The diffraction measure is the Fourier transform of γ , which has also been calculated in [34, 63]. Recalling $\widehat{\delta}_0 = \lambda$ and $\widehat{\lambda} = \delta_0$, the result is always of the form

$$(3.12) \quad \widehat{\gamma} = \delta_0 + (1 - b(k))\lambda = \delta_0 + h(k)\lambda,$$

where $b = \widehat{f}$. The Radon–Nikodym density h depends on β and is summarised in [14]. Figure 3 illustrates the result for the three ensembles.

A similar approach is possible on the basis of the eigenvalues of general complex random matrices. This leads to the ensemble studied by Ginibre [63], which is also discussed in [14]. One common feature of the resulting point sets is the effectively repulsive behaviour of the points, which leads to the ‘dip’ around 0 for $\hat{\gamma}$. For the two systems mentioned in this section, we omit the formulation of the full results and refer the reader to [14] for details. Further developments around determinantal and related point processes are described in reference [15].

4. THE RENEWAL PROCESS

A large and interesting class of processes in one dimension can be described as a *renewal* process [37, 4, 14]. Here, one starts from a probability measure μ on \mathbb{R}_+ (the positive real line) and considers a machine that moves at constant speed along the real line and drops a point on the line with a waiting time that is distributed according to μ . Whenever this happens, the internal clock is reset and the process resumes. Let us (for simplicity) assume that both the velocity of the machine and the expectation value of μ are 1, so that we end up with realisations that are, almost surely, point sets in \mathbb{R} of density 1 (after we let the starting point of the machine move to $-\infty$, say).

Clearly, the resulting process is stationary and can thus be analysed by considering all realisations which contain the origin. Moreover, there is a clear (distributional) symmetry around the origin, so that we can determine the corresponding autocorrelation γ of almost all realisations from studying what happens to the right of 0. Indeed, if we want to know the frequency per unit length of the occurrence of two points at distance x (or the corresponding density), we need to sum the contributions that x is the first point after 0, the second point, the third, and so on. In other words, we almost surely obtain the autocorrelation

$$(4.1) \quad \gamma = \delta_0 + \nu + \tilde{\nu}$$

with $\nu = \mu + \mu * \mu + \mu * \mu * \mu + \dots$, where the proper convergence of the sum of iterated convolutions follows from [4, Lemma 4] or from [12, Sec. 11.3]. Note that the point measure at 0 simply reflects the fact that the almost sure density of the resulting point set is 1. Indeed, ν is a translation bounded positive measure, and satisfies the renewal relations (compare [37, Ch. XI.9] or [4, Prop. 1] for a proof)

$$(4.2) \quad \nu = \mu + \mu * \nu \quad \text{and} \quad (1 - \hat{\mu}) \hat{\nu} = \hat{\mu},$$

where $\hat{\mu}$ is a uniformly continuous and bounded function on \mathbb{R} . The second equation emerges from the first by Fourier transform, but has been rearranged to highlight the relevance of the set $S = \{k \mid \hat{\mu}(k) = 1\}$ of singularities. In this setting, the measure γ of Eq. (4.1) is both positive and positive definite.

Based on the structure of the support of the underlying probability measure μ , one can determine the diffraction of the renewal process explicitly. To do so for a probability

measure μ on \mathbb{R}_+ with mean 1, we assume the existence of a moment of μ of order $1 + \varepsilon$ for some $\varepsilon > 0$; we refer to [4] for details on this condition. The diffraction measure of the point set realisations of the stationary renewal process based on μ almost surely is of the form

$$\widehat{\gamma} = \widehat{\gamma}_{\text{pp}} + (1 - h) \lambda,$$

where h is a locally integrable function on \mathbb{R} that is continuous almost everywhere. The pure point part is trivial, meaning $\widehat{\gamma} = \delta_0$, unless the support of μ is contained in a lattice. The details are stated below in Theorem 4.1. Proofs of these claims as well as further results can be found in [4, 14, 12].

The renewal process is a versatile method to produce interesting point sets on the line. These include random tilings with finitely many intervals (which are Delone sets) as well as the homogeneous Poisson process on the line (where μ is the exponential distribution with mean 1); see [4, Sec. 3] for explicit examples and applications. In particular, if one employs a suitably normalised version of the Gamma distribution, one can formulate a one-parameter family of renewal processes that continuously interpolates between the Poisson process (total positional randomness) and the lattice \mathbb{Z} (perfect periodic order); compare [4, Ex. 3] for more. The general result reads as follows.

Theorem 4.1. *Let ϱ be a probability measure on \mathbb{R}_+ with mean 1, and assume that a moment of ϱ of order $1 + \varepsilon$ exists for some $\varepsilon > 0$. Then, the point sets obtained from the stationary renewal process based on ϱ almost surely have a diffraction measure of the form*

$$\widehat{\gamma} = \widehat{\gamma}_{\text{pp}} + (1 - h) \lambda,$$

where h is a locally integrable function on \mathbb{R} that is continuous except for at most countably many points (namely those of the set $S = \{k \mid \widehat{\varrho}(k) = 1\}$). On $\mathbb{R} \setminus S$, the function h is given by

$$h(k) = \frac{2(|\widehat{\varrho}(k)|^2 - \text{Re}(\widehat{\varrho}(k)))}{|1 - \widehat{\varrho}(k)|^2}.$$

Moreover, the pure point part is

$$\widehat{\gamma}_{\text{pp}} = \begin{cases} \delta_0, & \text{if } \text{supp}(\varrho) \text{ is not a subset of a lattice,} \\ \delta_{\mathbb{Z}/b}, & \text{otherwise,} \end{cases}$$

where $b\mathbb{Z}$ is the coarsest lattice that contains $\text{supp}(\varrho)$. □

In one dimension, the renewal process allows an efficient derivation of the diffraction of random tilings, which we briefly summarise now.

5. RANDOM TILINGS

The deterministic Fibonacci chain can be defined by the primitive substitution rule $a \mapsto ab, b \mapsto a$, which defines a strictly ergodic (discrete) hull. When a and b are replaced

by intervals of length $\tau = \frac{1}{2}(1 + \sqrt{5})$ and 1, respectively, the left endpoints of the intervals define a model set (or cut and project set). The corresponding Dirac comb leads to the pure point diffraction measure

$$\widehat{\gamma}_F = \sum_{k \in \frac{1}{\sqrt{5}}\mathbb{Z}[\tau]} I(k) \delta_k$$

with intensities $I(k) = \left(\frac{\tau}{\sqrt{5}} \frac{\sin(\pi\tau k')}{\pi\tau k'}\right)^2$. Here, $\frac{\tau}{\sqrt{5}} = \frac{\tau+2}{5}$ is the density of the point set, and k' denotes the algebraic conjugate of k , which is defined on the field $\mathbb{Q}(\sqrt{5})$ by $\sqrt{5} \mapsto -\sqrt{5}$ and acts as the \star -map for the underlying model set description. In particular, the diffraction is the same for all Dirac combs of the Fibonacci hull; see [12, Sec. 9.4.1] and references therein for details. An illustration is shown in the upper panel of Figure 4.

The corresponding random tiling ensemble consists of all tilings of the real line by the two types of intervals. For a direct comparison, it makes more sense to only consider those tilings with the same relative frequency of interval types, which means frequencies $1/\tau$ and $1/\tau^2$ for the long and the short interval, respectively.

The diffraction of a typical Dirac comb out of this class was originally derived in [13], but can also be obtained via an application of the renewal structure from Theorem 4.1. This leads to

$$\widehat{\gamma}_{\text{rt}} = \frac{\tau+1}{5} \delta_0 + h\lambda$$

with the Radon–Nikodym density

$$h(k) = \frac{\tau+2}{5} \frac{(\sin(\pi k/\tau))^2}{\tau^2(\sin(\pi k\tau))^2 + \tau(\sin(\pi k))^2 - (\sin(\pi k/\tau))^2}.$$

Except for the trivial Bragg peak at $k = 0$, the diffraction measure is thus absolutely continuous. Still, the resemblance between this function and the diffraction of the perfect Fibonacci chain is remarkable, as can be seen from Figure 4.

The situation in dimensions $d \geq 2$ is less favourable from a mathematical perspective, although one has a rather clear intuition of what one should expect [43, 70], based on solid scaling arguments. In dimensions $d \geq 3$, a mixed spectrum with pure point and absolutely continuous components is conjectured, while $d = 2$ is the critical dimension in the sense that random tilings with non-crystallographic symmetries should display a singular continuous component; see [12, Sec. 11.6.2] for an example.

Unfortunately, only few results have been proved so far. Among them are a rigorous treatment of planar random tiling ensembles with crystallographic symmetries (such as the lozenge tiling and several relatives, see [13, 45, 46]), a group-theoretic approach to one of the random tiling hypotheses [69, 70] and a treatment of dense Dirac combs with pure point diffraction [71, 59] that is needed to understand the pure point part of the random tiling diffraction in dimensions $d \geq 3$. The remaining questions are still open, though there is little doubt that the original analysis from [43] is essentially correct.

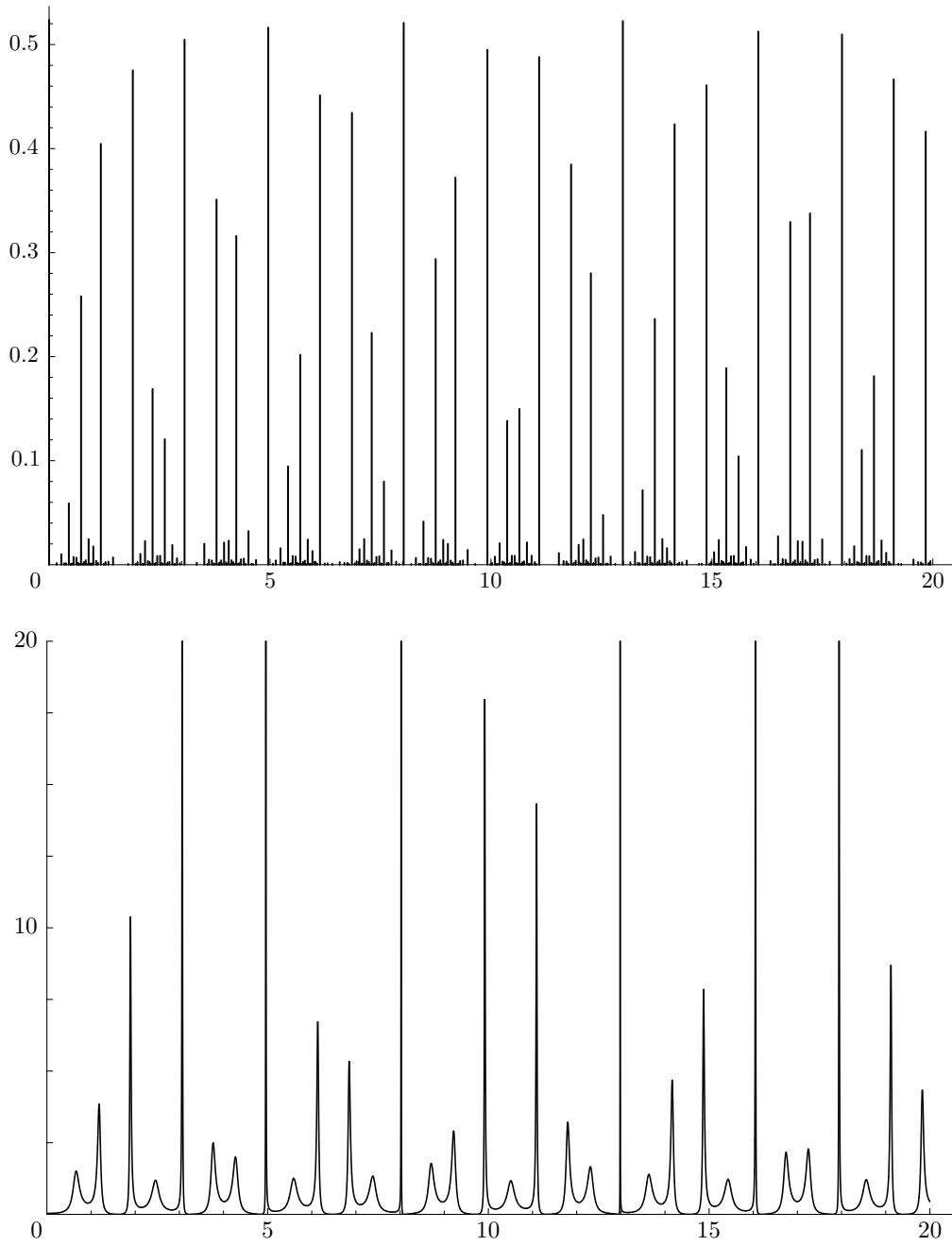


FIGURE 4. The pure point diffraction measure of the perfect Fibonacci chain (upper panel) and the absolutely continuous part of the corresponding random tiling (lower panel). Bragg peaks (in the upper picture) are shown as lines, where the height is the intensity, while the smooth Radon–Nikodym density in the lower picture is truncated at a value of 20 to illustrate the spikyness. The central peak (of intensity $\frac{\tau+1}{5}$) is omitted in both diagrams.

Let us now leave the realm of explicit examples and turn our attention to a more general approach of systems with randomness, formulated with methods from the theory of point processes; compare [42, 57, 58, 4] for related approaches and results.

6. STOCHASTIC POINT PROCESSES AND THE PALM MEASURE

In this section, we take the viewpoint of a general shift-invariant random measure and relate its realisation-wise diffraction to its second moment measure. As such, this section is a complex-valued extension of [4, Sec. 5].

Let $\mu = \mu_{\Re} + i\mu_{\Im}$ be a locally finite complex-valued measure on \mathbb{R}^d (which means that μ_{\Re} and μ_{\Im} are both locally finite signed measures). A short calculation reveals that, for $f \in C_c(\mathbb{R}^d, \mathbb{C})$ of the form $f = g + ih$ with real-valued g and h , the measure $\tilde{\mu}$ can consistently be defined via

$$\tilde{\mu}(f) := \overline{\mu(\tilde{f})} = \overline{\mu}(f_-) = \mu_{\Re}(f_-) - i\mu_{\Im}(f_-),$$

where $\tilde{f}(x) = \overline{f_-(-x)}$ with $f_-(x) = f(-x)$. In particular, note that

$$\overline{\mu} = \mu_{\Re} - i\mu_{\Im} \quad \text{and} \quad \overline{\mu(f)} = \overline{\mu}(\tilde{f})$$

hold as expected. The point here is that, after having dealt with the case of real (or signed) measures, the extension to complex measures is canonical and consistent.

To continue, recall the *polar representation* of a complex measure from [33, Ch. XIII.16]; see also [12, Prop. 8.3]. Given μ , there is a measurable function $\alpha_{\mu}: \mathbb{R}^d \rightarrow [0, 2\pi)$ such that, for $f \in C_c(\mathbb{R}^d, \mathbb{C})$, one has

$$\int_{\mathbb{R}^d} f(x) d\mu(x) = \int_{\mathbb{R}^d} f(x) e^{i\alpha_{\mu}(x)} d|\mu|(x),$$

where $|\mu|$ is the total variation measure of μ . This means that $|\mu|$ is the smallest non-negative measure such that $|\mu(A)| \leq |\mu|(A)$ for any bounded and measurable A , where $|\mu| \leq |\mu_{\Re}| + |\mu_{\Im}|$; compare [12, Sec. 8.5.1] and references therein.

Let \mathcal{M} denote the \mathbb{C} -vector space of all locally finite, complex-valued measures ϕ on \mathbb{R}^d , so $\phi \in \mathcal{M}$ means $|\phi(A)| < \infty$ for any bounded Borel set A . A sequence $(\phi_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ converges vaguely to ϕ if $\phi_n(f) \rightarrow \phi(f)$ as $n \rightarrow \infty$ for all $f \in C_c(\mathbb{R}^d)$. The space \mathcal{M} is closed in the topology of vague convergence of measures (in fact, \mathcal{M} is a Polish space with this topology). We let $\Sigma_{\mathcal{M}}$ denote the σ -algebra of Borel sets of \mathcal{M} . The latter can be described as the σ -algebra of subsets of \mathcal{M} generated by the requirement that, for all bounded Borel sets $A \subset \mathbb{R}^d$, the mapping $\phi \mapsto \phi(A)$ is measurable.

For each $t \in \mathbb{R}^d$, let T_t denote the translation operator on \mathbb{R}^d , as defined by the mapping $x \mapsto t + x$. Clearly, $T_t T_s = T_{t+s}$, and the inverse of T_t is given by $T_t^{-1} = T_{-t}$. For functions f on \mathbb{R}^d , the corresponding translation action is defined via $T_t f = f \circ T_{-t}$, so that $(T_t f)(x) = f(x - t)$. Similarly, for $\phi \in \mathcal{M}$, let $T_x \phi := \phi \circ T_{-x}$ be the image measure under the translation, so that $(T_x \phi)(A) = \phi(T_{-x}(A)) = \phi(A - x)$ for any measurable

subset $A \subset \mathbb{R}^d$, and $(T_x\phi)(f) = \int_{\mathbb{R}^d} f(y) d(T_x\phi)(y) = \int_{\mathbb{R}^d} f(x+z) d\phi(z) = \phi(T_{-x}f)$ for functions. This means that there is a translation action of \mathbb{R}^d on \mathcal{M} . Finally, we also have a translation action on $\mathcal{P}(\mathcal{M})$, the probability measures on \mathcal{M} , via $(T_xQ)(A) = Q(T_{-x}A)$ for any $A \in \Sigma_{\mathcal{M}}$ and $Q \in \mathcal{P}(\mathcal{M})$. A set $A \in \Sigma_{\mathcal{M}}$ is called *invariant* (under translations) if $T_{-x}A = A$ for all $x \in \mathbb{R}^d$.

A (complex-valued) *random measure* Φ is a random variable (defined on some probability space $(\Theta, \mathcal{F}, \pi)$) with values in \mathcal{M} , which formally means that $\Phi: \Theta \rightarrow \mathcal{M}$ is an $(\mathcal{F}-\Sigma_{\mathcal{M}})$ -measurable function. Its distribution is then $Q = \pi \circ \Phi^{-1} \in \mathcal{P}(\mathcal{M})$, i.e. the image measure of π under Φ . We will follow the usual practice in probability theory and not make the underlying probability space explicit (a canonical choice can in many cases simply be $\Theta = \mathcal{M}$ and $\Phi = \text{Id}_{\mathcal{M}}$). We will also usually suppress the dependence of Φ on $\theta \in \Theta$ in the notation. Integrals over Θ w.r.t. the probability measure π will be denoted by \mathbb{E} , the expectation value.

Φ is called *stationary* if its distribution Q satisfies $T_xQ = Q$ for all $x \in \mathbb{R}^d$. A stationary random measure is called *ergodic* if the shift-invariant σ -algebra is trivial, which means that any invariant A has probability 0 or 1 (more generally, one requires $Q(A) \in \{0, 1\}$ whenever $Q((T_{-x}A) \triangle A) = 0$ for all $x \in \mathbb{R}^d$; compare [30, Def. 10.3.I and Prop. 10.3.III]).

In what follows, we generally assume that

$$(6.1) \quad \Phi \text{ is a (possibly) complex-valued, stationary and ergodic random measure on } \mathbb{R}^d,$$

which means that there is a decomposition $\Phi = \Phi_{\Re} + i\Phi_{\Im}$ where both Φ_{\Re} and Φ_{\Im} are signed, real-valued, stationary, ergodic random measures on \mathbb{R}^d . To verify the last statement note that, since for any bounded measurable $A \subset \mathbb{R}^d$, $\theta \mapsto \Phi(A) (= \Phi(\theta, A)) \in \mathbb{C}$ is measurable, also $\Phi_{\Re}(A)$ and $\Phi_{\Im}(A)$ are measurable as functions of θ . Consider any shift-invariant measurable $B \subset \mathcal{M}_{\text{real}}$ ($\mathcal{M}_{\text{real}}$ denotes the locally finite signed measures on \mathbb{R}^d), then $\{\Phi \mid \Phi_{\Re} \in B\}$ is shift invariant and measurable as well, so $\mathbb{P}(\Phi_{\Re} \in B) \in \{0, 1\}$, and analogously for Φ_{\Im} . We further assume that Φ is locally square integrable in the sense that

$$(6.2) \quad \mathbb{E} \left[(|\Phi_{\Re}|(A))^2 + (|\Phi_{\Im}|(A))^2 \right] < \infty \quad \text{for all bounded } A \subset \mathbb{R}^d,$$

where $|\Phi_{\Re}|$ and $|\Phi_{\Im}|$ denote the total variation measures of Φ_{\Re} and Φ_{\Im} , respectively.

In analogy with the real-valued case in [4, Sec. 5.2], we define $\mu^{(2)}$, the second moment measure of Φ , via

$$(6.3) \quad \mu^{(2)}(A \times A') = \mathbb{E}[\Phi(A)\overline{\Phi(A')}] \quad \text{for bounded } A, A' \in \mathcal{B}(\mathbb{R}^d),$$

hence, for $f \in C_c(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$,

$$\mu^{(2)}(f) = \mathbb{E} \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) d\Phi(x) d\overline{\Phi}(y) \right].$$

By the shift invariance of the distribution of Φ , we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x + z, y + z) d\mu^{(2)}(x, y)$$

for all $z \in \mathbb{R}^d$, and hence we can factor out this symmetry to obtain the reduced second moment measure $\mu_{\text{red}}^{(2)}$. The latter is a locally finite complex-valued measure that is characterised by

$$(6.4) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u + v, u) d\mu_{\text{red}}^{(2)}(v) d\lambda(u)$$

for $f \in C_c(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$. By the shift invariance of Lebesgue measure on \mathbb{R}^d , we equivalently have

$$(6.5) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u, u - v) d\mu_{\text{red}}^{(2)}(v) d\lambda(u).$$

To prove the existence of $\mu_{\text{red}}^{(2)}$, one can decompose $\mu^{(2)} = \mu_{\Re}^{(2)} + i\mu_{\Im}^{(2)}$ into real and imaginary parts and then use the well-known real-valued results (compare [30, Lemma 10.4.III]) to obtain $\mu_{\text{red}}^{(2)} = \mu_{\Re, \text{red}}^{(2)} + i\mu_{\Im, \text{red}}^{(2)}$.

Note that $\mu_{\text{red}}^{(2)}$ is uniquely defined and is a positive definite measure, since

$$(6.6) \quad \begin{aligned} \mu^{(2)}(f \otimes \bar{g}) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u + v) \overline{g(u)} d\mu_{\text{red}}^{(2)}(v) d\lambda(u) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v - w) \overline{g(-w)} d\lambda(w) d\mu_{\text{red}}^{(2)}(v) \\ &= \int_{\mathbb{R}^d} (f * \tilde{g})(v) d\mu_{\text{red}}^{(2)}(v) = \mu_{\text{red}}^{(2)}(f * \tilde{g}), \end{aligned}$$

so that

$$\begin{aligned} \mu_{\text{red}}^{(2)}(f * \tilde{f}) &= \mu^{(2)}(f \otimes \bar{f}) = \mathbb{E} \left[\int f d\Phi \int \bar{f} d\bar{\Phi} \right] \\ &= \mathbb{E} \left[\int f d\Phi \overline{\int f d\Phi} \right] = \mathbb{E} [| \Phi(f) |^2] \geq 0. \end{aligned}$$

Remark 6.1 (see also [4, Rem. 13]). One can alternatively define

$$\mu^{(2, \text{alt})}(A \times A') = \mathbb{E} [\overline{\Phi(A)} \Phi(A')] \quad \text{for bounded } A, A' \in \mathcal{B}(\mathbb{R}^d),$$

and then obtain $\mu_{\text{red}}^{(2, \text{alt})}$ from this as above, via

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2, \text{alt})}(x, y) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u + v, u) d\mu_{\text{red}}^{(2, \text{alt})}(v) d\lambda(u) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u, u - v) d\mu_{\text{red}}^{(2, \text{alt})}(v) d\lambda(u). \end{aligned}$$

Then, we have $\mu^{(2,\text{alt})} = \overline{\mu^{(2)}}$ and $\mu_{\text{red}}^{(2,\text{alt})} = \overline{\mu_{\text{red}}^{(2)}}$. Since

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(y, x) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\overline{\mu^{(2)}}(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2,\text{alt})}(x, y),$$

we see that the alternative choice of factoring out the shift invariance in Eq. (6.4), namely integrating $f(u, u + v)$ on the right-hand side of this equation, leads to $\mu_{\text{red}}^{(2,\text{alt})}$, where

$$(6.7) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) d\mu^{(2)}(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u, u + v) d\mu_{\text{red}}^{(2,\text{alt})}(v) d\lambda(u).$$

We choose the definitions as in Eqs. (6.3) and (6.4) because these fit well to the formulation of the limit in Eq. (6.8) below. Note that, in the real-valued case, $\mu_{\text{red}}^{(2)}$ and $\mu_{\text{red}}^{(2,\text{alt})}$ agree.

The ‘complex-valued’ analogue of [4, Thm. 5] now reads as follows.

Theorem 6.2. *Assume that conditions (6.1) and (6.2) are satisfied, and let $\Phi_n := \Phi|_{B_n}$ denote the restriction of Φ to the open ball of radius n around 0. Then, the natural autocorrelation of Φ , which is defined with an averaging sequence of nested, centred balls, almost surely exists and satisfies*

$$(6.8) \quad \gamma^{(\Phi)} := \lim_{n \rightarrow \infty} \frac{\Phi_n * \widetilde{\Phi_n}}{\lambda(B_n)} = \lim_{n \rightarrow \infty} \frac{\Phi_n * \widetilde{\Phi}}{\lambda(B_n)} = \mu_{\text{red}}^{(2)},$$

where the limit refers to the vague topology. In particular, the autocorrelation is non-random.

Proof. The proof is a suitable ‘complex-valued interpretation’ of the proof of [4, Thm. 5]. Fix a continuous function $f: \mathbb{R}^d \rightarrow \mathbb{C}$ with compact support. We have to check that

$$(6.9) \quad \frac{1}{\lambda(B_n)} (\Phi_n * \widetilde{\Phi_n})(f) \xrightarrow{n \rightarrow \infty} \mu_{\text{red}}^{(2)}(f) \quad (\text{a.s.}).$$

Since both sides are locally finite (complex-valued) measures, it actually suffices to check Eq. (6.9) for real-valued f . For $x \in \mathbb{R}^d$, define

$$F(x) := \int_{\mathbb{R}^d} f(x - y) d\overline{\Phi}(y) = \int_{\mathbb{R}^d} f(x + y) d\widetilde{\Phi}(y).$$

Clearly, F inherits stationarity and ergodicity from Φ , wherefore F is a (complex-valued) ergodic random function on \mathbb{R}^d in the sense that shift-invariant events for F have ‘trivial’ probabilities (0 or 1), and we obtain

$$\mathbb{E} \left[\int_A |F(x)| d|\Phi|(x) \right] < \infty$$

for any bounded and measurable $A \subset \mathbb{R}^d$.

Define a (complex-valued) additive covariant spatial process $X(A)$ in the sense of [66], indexed by a bounded and measurable $A \subset \mathbb{R}^d$, via

$$X(A) := \int_A F(x) d\Phi(x).$$

Covariant in this context means that X behaves ‘naturally’ under translations: When \mathbb{R}^d acts on X via $(T_u X)(A) := \int_A F(x) d(T_u \Phi)(x)$, for $u \in \mathbb{R}^d$, then $(T_u X)(A + u) = X(A)$.

Decomposing X into its real and imaginary parts (by decomposing F and Φ and suitably grouping terms) we can apply [66, Cor. 4.9] to obtain a.s.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} (\Phi_n * \tilde{\Phi})(f) &= \lim_{n \rightarrow \infty} \frac{1}{\lambda(B_n)} \int_{B_n} F(x) d\Phi(x) = \lim_{n \rightarrow \infty} \frac{X(B_n)}{\lambda(B_n)} \\ &= \mathbb{E} \left[\frac{X(B_1)}{\lambda(B_1)} \right] = \frac{1}{\lambda(B_1)} \mathbb{E} \left[\int_{B_1} \int_{\mathbb{R}^d} f(x-y) d\bar{\Phi}(y) d\Phi(x) \right] \\ &= \frac{1}{\lambda(B_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{B_1}(x) f(x-y) d\mu^{(2)}(x, y) \\ &= \frac{1}{\lambda(B_1)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x) f(z) d\mu_{\text{red}}^{(2)}(z) d\lambda(x) = \int_{\mathbb{R}^d} f d\mu_{\text{red}}^{(2)}. \end{aligned}$$

The difference between $\Phi_n * \tilde{\Phi}$ and $\Phi_n * \tilde{\Phi}_n$ is a (random) ‘boundary term’ that almost surely vanishes in the limit as $n \rightarrow \infty$. To prove this formally, decompose $\Phi = \Phi_{\Re} + i\Phi_{\Im}$, $\tilde{\Phi} = \tilde{\Phi}_{\Re} - i\tilde{\Phi}_{\Im}$ and then argue as in the proof of [4, Thm. 3] for each of the four terms appearing in $\Phi_n * (\tilde{\Phi} - \tilde{\Phi}_n)$. \square

Remark 6.3. Theorem 6.2 allows to reformulate Theorem 4 and Corollary 1 from [4] for complex-valued clusters as follows. If Φ is a stationary ergodic point process, i.e. Φ is a random sum of Dirac measures, with distribution P satisfying Eq. (6.2), and if we replace each point independently by a random complex-valued measure with distribution Q , then the formulas describing the autocorrelation and the diffraction of the resulting cluster process given in [4, Thm. 4 and Cor. 1] continue to hold.

Let us also mention that, by specialising Φ to a renewal process, Theorem 6.2 allows to recover Eq. (4.1) and, in particular, Theorem 4.1 from this more general perspective; see [4] for further details, and how this can be used to formulate the renewal process also for more general ‘dropping’ distributions.

6.1. A ‘Palm-type Distribution’ for Complex-valued Random Measures. In the case of a positive random measure Φ , Eq. (6.8) can be interpreted via the Palm distribution P_0 of the law of Φ , which is a probability measure on locally finite measures (intuitively, the law of Φ viewed relative to a typical point of its support) via

$$(6.10) \quad \mu_{\text{red}}^{(2)} = \rho I_{P_0}$$

where $\rho > 0$ is the intensity and I_{P_0} the first moment measure of P_0 ; compare [4, Sec. 5.2]. This interpretation breaks down in general in the signed or complex-valued case because $\mu_{\text{red}}^{(2)}$ will not be a positive measure. One way to extend this line of thought is to re-interpret the Palm distributions in a way suited for complex-valued random measures as follows.

Recalling the structure of the polar decomposition, the random measure Φ can equivalently be described via $(|\Phi|, \Phi_{\text{ph}})$, where $|\Phi|$ is the total variation measure and the mapping $\Phi_{\text{ph}}: \mathbb{R}^d \rightarrow [0, 2\pi)$ the ‘phase function’:

$$(6.11) \quad \int_{\mathbb{R}^d} f(x) d\Phi(x) = \int_{\mathbb{R}^d} f(x) e^{i\Phi_{\text{ph}}(x)} d|\Phi|(x).$$

Note that $\Phi \mapsto (|\Phi|, \Phi_{\text{ph}})$ is measurable, so $(|\Phi|, \Phi_{\text{ph}})$ is in fact a random variable. Define a positive σ -finite measure \mathcal{C} on $\mathbb{R}^d \times \mathcal{M}$ (this is the equivalent of the so-called Campbell measure for the complex-valued context and agrees with the usual Campbell measure if Φ is a positive random measure) via

$$\int_{\mathbb{R}^d \times \mathcal{M}} g(x, \varphi) d\mathcal{C}(x, \varphi) := \mathbb{E} \left[\int_{\mathbb{R}^d} g(x, e^{-i\Phi_{\text{ph}}(x)} \Phi) d|\Phi|(x) \right],$$

whenever the right-hand side is defined (which will for instance always be the case when g is measurable and non-negative). By the shift invariance of Φ , and hence that of $|\Phi|$, the projection of \mathcal{C} to \mathbb{R}^d is ρ times Lebesgue measure (with $\rho \in [0, \infty)$ being the intensity of $|\Phi|$), hence there is a family of probability measures P_x on \mathcal{M} , with $P_x \in \mathcal{P}(\mathcal{M})$ for all $x \in \mathbb{R}^d$, so that we can disintegrate (compare [48, Thm. 15.3.3])

$$(6.12) \quad \int g d\mathcal{C} = \int_{\mathbb{R}^d} \int_{\mathcal{P}(\mathcal{M})} g(x, \varphi) dP_x(\varphi) \rho d\lambda(x).$$

Definition 6.4. We call the elements of the family $\{P_x \mid x \in \mathbb{R}^d\}$ the *Palm distributions* in the complex-valued case.

Let, for $A \subset \mathbb{R}^d$ bounded and measurable,

$$I_{P_x}(A) := \int_{\mathcal{M}} \varphi(A) dP_x(\varphi)$$

be the expectation (or first moment) measure of P_x . By shift invariance, we have $P_x = T_x P_0$, $x \in \mathbb{R}^d$, and hence $I_{P_x} = T_x I_{P_0}$. The connection between the (reduced) second moment measure and the Palm distribution carries over to the complex-valued case as follows.

Proposition 6.5. *For the extended definition of the Palm distribution, one has*

$$\mu_{\text{red}}^{(2)} = \rho I_{P_0},$$

so Eq. (6.10) also holds in this case.

Sketch of Proof. Consider $g(x, \varphi) = \mathbf{1}_{A'}(x)\varphi(A)$ with $\mathbf{1}$ denoting the characteristic function and with $A, A' \subset \mathbb{R}^d$ bounded and measurable. Then,

$$(6.13) \quad \begin{aligned} \int g \, d\mathcal{C} &= \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{1}_{A'}(x) e^{-i\Phi_{\text{ph}}(x)} \Phi(A) \, d|\Phi|(x) \right] \\ &= \mathbb{E} \left[\Phi(A) \int_{\mathbb{R}^d} \mathbf{1}_{A'}(x) e^{-i\Phi_{\text{ph}}(x)} \, d|\Phi|(x) \right] = \mathbb{E} [\Phi(A) \overline{\Phi(A')}] \end{aligned}$$

by definition, whereas the disintegration formula yields

$$(6.14) \quad \begin{aligned} \int g \, d\mathcal{C} &= \int_{\mathbb{R}^d} \int_{\mathcal{M}} \mathbf{1}_{A'}(x) \varphi(A) \, dP_x(\varphi) \rho \, d\lambda(x) = \rho \int_{A'} I_{P_x}(A) \, d\lambda(x) \\ &= \rho \int_{A'} I_{P_0}(A - x) \, d\lambda(x) = \rho \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{A'}(x) \mathbf{1}_{A-x}(y) \, dI_{P_0}(y) \, d\lambda(x) \\ &= \rho \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{A'}(x) \mathbf{1}_A(y + x) \, d\lambda(x) \, dI_{P_0}(y) \\ &= \rho \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{A'}(-x) \mathbf{1}_A(y - x) \, d\lambda(x) \, dI_{P_0}(y) = \rho I_{P_0}(\mathbf{1}_A * \widetilde{\mathbf{1}_{A'}}). \end{aligned}$$

Comparing Eqs. (6.13)–(6.14) with Eq. (6.6) yields the claim. \square

If Φ is ergodic, the viewpoint that P_0 describes the configuration relative to a point in the support drawn according to Φ is corroborated by

$$\frac{1}{\lambda(B_n)} \int_{B_n} g(e^{-i\Phi_{\text{ph}}(x)} T_{-x}\Phi) \, d|\Phi|(x) \xrightarrow{n \rightarrow \infty} \int_{\mathcal{M}} g(\varphi) \, dP_0(\varphi) \quad (\text{a.s.})$$

for any bounded measurable $g: \mathcal{M} \rightarrow \mathbb{R}$.

The viewpoint of (possibly complex-valued) ergodic random measures for diffraction is a useful one since it provides a connection to the large literature on random measures and on stochastic geometry; see [30, 49, 51, 52, 62, 4] and references therein, as well as [75] for a recent generalisation that can also be considered from the diffraction point of view. However, our approach also shows a limitation that one encounters when trying to infer properties of a random configuration of scatterers from its kinematic diffraction: As is evident from Eq. (6.8) in Theorem 6.2, the only ‘datum’ from a random Φ visible in its autocorrelation, and hence also in the corresponding diffraction, is the second moment measure. It is well known that second moments are generally insufficient to determine the distribution of Φ unless further structural properties are known. This inverse problem is known as the homometry problem in crystallography and the inference problem in the theory of stochastic processes.

7. OUTLOOK

Our exposition provides a snapshot of the present knowledge about systems with continuous diffraction components; see [12, Chs. 10 and 11] as well as [4, 15] for additional examples, and [20, 15] for connections with the dynamical spectrum. Nevertheless, as is apparent from a comparison with the pure point diffraction case [23, 12, 58, 77, 76], the status of general results is lagging behind. Even for many important examples, some of the most obvious questions are still open from a mathematical point of view. In particular, this is so for random tiling ensembles in dimensions $d \geq 2$, or for equilibrium systems just beyond the complexity of the (planar) Ising model.

Apart from the systems considered here, an interesting class is provided by random substitution and inflation systems, as introduced in [41]. The randomness present here is compatible with the long-range order of Meyer sets with entropy [21, 64, 65], which means that one obtains interesting mixtures of pure point and absolutely continuous diffraction measures. Though this direction has not attracted much attention so far, it is both tractable and practically relevant.

From a more general perspective, one lacks some kind of analogue to the key theorems in pure point diffraction (such as the Poisson summation formula or the Halmos–von Neumann theorem). While there is at least the theory of Riesz products [80, 68] for self-similar systems with singular spectra, a general approach to stochastic systems is only at its beginning. Methods from point process theory [30], such as the Palm measure and its connection to the autocorrelation (via its intensity measure), look promising, but have not produced many concrete results so far. The latter, however, are needed to make some progress with the complicated inverse problem for such systems. Though there is substantial knowledge from the inference approach [49], it is not clear at present how this can be used, and how reasonable restrictions could be included.

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